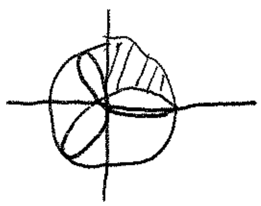
Exam 3

Math 233

This exam consists of 12 questions worth 8 points each. **You must show all work. Answers without work will receive no credit.** All answers should be exact–do not approximate unless explicitly instructed.

1. Find the area between the three-petaled rose $r = \cos 3\theta$ and the circle $r = 1$ in the first quadrant. You may want to think carefully about how the rose is traced out as θ increases.



Area = Area of 1/4 circle − Area of rose in Q1

$$= \frac{1}{4}\pi - \int_0^{\frac{\pi}{6}} \int_0^{\cos 3\theta} r\,dr\,d\theta$$

$$= \frac{1}{4}\pi - \int_0^{\frac{\pi}{6}} \frac{1}{2} r^2 \Big|_{r=0}^{r=\cos 3\theta} d\theta$$

$$= \frac{1}{4}\pi - \frac{1}{2}\int_0^{\frac{\pi}{6}} \cos^2(3\theta)\,d\theta$$

$u = 3\theta$

$\frac{du}{3} = d\theta$

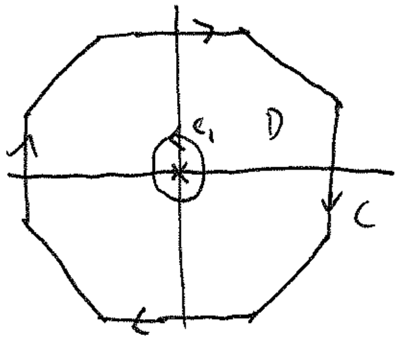
$$= \frac{1}{4}\pi - \frac{1}{6}\int_0^{\frac{\pi}{2}} \cos^2 u\,du$$

$$= \frac{1}{4}\pi - \frac{1}{6}\left(\frac{1}{2}u + \frac{1}{4}\sin 2u\right)\Big|_{u=0}^{u=\frac{\pi}{2}}$$

$$= \frac{1}{4}\pi - \frac{1}{6}\left(\frac{\pi}{4}\right)$$

$$= \frac{5\pi}{24}$$

2. Let $\mathbf{F} = \left(\overset{P}{\frac{-y}{x^2+y^2}}, \overset{Q}{\frac{x}{x^2+y^2}}\right)$ be the vortex vector field, and let C be a regular octagon of sidelength 10, with four sides orthoganal to the coordinate axes, oriented **clockwise**. Compute $\oint_C \mathbf{F} \cdot d\mathbf{s}$.



Let C_1 be the unit circle oriented counter clockwise and D region between C and C_1.

We've computed several times $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 0$

$$0 = \iint_D 0\, dA = \iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\, dA$$

$$= \oint_{\partial D} \vec{F} \cdot d\vec{s} \qquad \text{by Green's Theorem}$$

$$= \oint_{-C} \vec{F} \cdot d\vec{s} + \oint_{-C_1} \vec{F} \cdot d\vec{s}$$

$$= -\oint_C \vec{F} \cdot d\vec{s} + -\int_{C_1} \vec{F} \cdot d\vec{s}$$

C_1 has parameterization

$\vec{r}(t) = (\cos t, \sin t)$

$$\oint_C \vec{F} \cdot d\vec{s} = -\int_{C_1} \vec{F} \cdot d\vec{s}$$

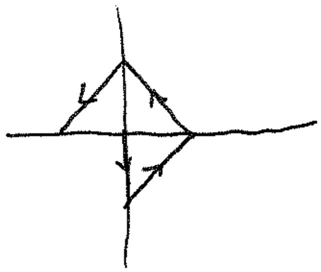
$$= -\int_0^{2\pi} \left(\frac{-\sin t}{\cos^2 t + \sin^2 t}, \frac{\cos t}{\cos^2 t + \sin^2 t}\right) \cdot (-\sin t, \cos t)\, dt$$

$$= -\int_0^{2\pi} \sin^2 t + \cos^2 t\, dt$$

$$= -\int_0^{2\pi} dt$$

$$= -2\pi$$

3. Let C be the piecewise linear curve from $(0,0)$ to $(0,-1)$ to $(1,0)$ to $(0,1)$ to $(-1,0)$. Compute $\int_C 2xy\,dx + (x^2+2x)\,dy$.



Close off path by C_1 from $(-1,0)$ to $(0,0)$ parameterized $\vec{r}(t) = (t,0)$ $-1 \le t \le 0$

D is $\frac{3}{4}$ of a square of side length $\sqrt{2}$, so Area$(D) = \frac{3}{4}(\sqrt{2})^2 = \frac{3}{2}$

$$\int_C 2xy\,dx + (x^2+2x)\,dy + \int_{C_1} 2xy\,dx + (x^2+2x)\,dy$$
$$= \iint_D (2x+2) - 2x \, dA$$
$$= 2 \iint_D dA$$
$$= 2\left(\frac{3}{2}\right)$$
$$= 3$$

$$\int_C 2xy\,dx + (x^2+2x)\,dy = 3 - \int_{C_1} 2xy\,dx + (x^2+2x)\,dy$$
$$= 3 - \int_{-1}^{0} 2(t)(0)(1) + (t^2+2t)(0)\,dt$$
$$= 3 - \int_{-1}^{0} 0 \, dt$$
$$= 3$$

4. Let $\mathbf{r}(t) = \left(\frac{3t}{1+t^3}, \frac{3t^2}{1+t^3}\right)$. The portion of the curve $0 \le t \le \infty$ forms a simple closed curve (i.e. a loop) in the first quadrant starting and ending at the origin. Find the area enclosed by this loop.

$$\text{Area} = \iint_D dA = \frac{1}{2}\oint_{\partial D} -y\,dx + x\,dy \qquad \text{(Green's Thm)}$$

$$= \frac{1}{2}\int_{\vec{r}} -y\,dx + x\,dy$$

$$= \frac{1}{2}\int_0^\infty -\left(\frac{3t^2}{(1+t^3)}\right)\cdot\left(\frac{3(1+t^3)-3t^2(3t)}{(1+t^3)^2}\right) + \left(\frac{3t}{(1+t^3)}\right)\cdot\left(\frac{6t(1+t^3)-3t^2(3t^2)}{(1+t^3)^2}\right)dt$$

$$= \frac{1}{2}\int_0^\infty \frac{3t}{(1+t^3)^3}\left[(-3t-3t^4+9t^4) + (6t+6t^4-9t^4)\right]dt$$

$$= \frac{1}{2}\int_0^\infty \frac{3t}{(1+t^3)^3}\,(3t+3t^4)\,dt$$

$$= \frac{1}{2}\int_0^\infty \frac{3t}{(1+t^3)^3}\cdot 3\cdot t\cdot(1+t^3)\,dt$$

$$= \frac{3}{2}\int_0^\infty \frac{3t^2}{(1+t^3)^2}\,dt$$

$u = 1+t^3$

$du = 3t^2\,dt$

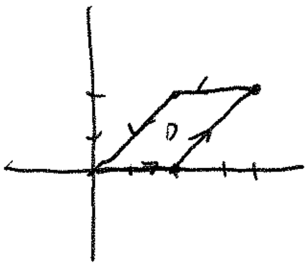
$$= \frac{3}{2}\int_1^\infty \frac{1}{u^2}\,du$$

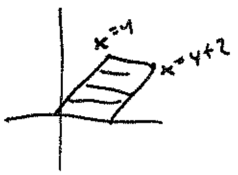
$$= \frac{3}{2}\left(-\frac{1}{u}\Big|_{u=1}^{u=\infty}\right)$$

$$= \frac{3}{2}\left(\lim_{a\to\infty} -\frac{1}{a} + 1\right)$$

$$= \frac{3}{2}$$

5. Let C be the boundary of the parallelogram with corners at $(0,0)$, $(2,0)$, $(4,2)$, and $(2,2)$ oriented counter clockwise. Compute $\oint_C 2xy\ dx - 7y\ dy$.





$$\oint_{\partial D} 2xy\,dx - 7y\,dy = \iint_D (0 - 2x)\,dA$$

$$= \int_0^2 \int_y^{y+2} -2x \, dx\, dy$$

$$= \int_0^2 -x^2 \Big|_{x=y}^{x=y+2} dy$$

$$= \int_0^2 -(y+2)^2 + y^2 \, dy$$

$$= \int_0^2 -4y - 4 \quad dy$$

$$= -2y^2 - 4y \Big|_{y=0}^{y=2}$$

$$= -2(4) - 4(2)$$

$$= -16$$

6. Find a parameterization for the bottom sheet of the two-sheeted hyperboloid $x^2 + y^2 = z^2 - 1$.

Convert to Cylindrical

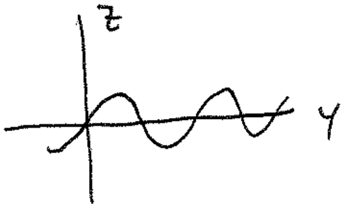
$$r^2 = z^2 - 1$$

$$z^2 = r^2 + 1$$

$$z = -\sqrt{r^2+1}$$ (negative gives us bottom sheet)

$$\Phi = \left(r\cos\theta,\ r\sin\theta,\ -\sqrt{r^2+1} \right)$$

7. Find a parameterization for the surface obtained by rotating the curve $z = \sin y$ about the y-axis.



$\vec{r}(t) = (0,\ y,\ \sin y)$

about y-axis

$\Phi = ((\sin y)\cos\theta,\ y,\ (\sin y)\sin\theta)$

8. Compute $\iint_S 2x - 2z dS$ where S is the surface parameterized by $\Phi = (\frac{u^2}{2}, uv, v^2)$ with $0 \le u, v \le \sqrt{5}$.

$$\frac{\partial \Phi}{\partial u} = (u, v, 0)$$

$$\frac{\partial \Phi}{\partial v} = (0, u, 2v)$$

$$\vec{n} = \frac{\partial \Phi}{\partial u} \times \frac{\partial \Phi}{\partial v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ u & v & 0 \\ 0 & u & 2v \end{vmatrix} = (2v^2, -2uv, u^2)$$

$$\|\vec{n}\| = \sqrt{4v^4 + 4u^2v^2 + u^4} = \sqrt{(2v^2+u^2)^2} = u^2 + 2v^2$$

$$\iint_S 2x - 2z dS = \int_0^{\sqrt{5}} \int_0^{\sqrt{5}} \left(2\left(\frac{u^2}{2}\right) - 2(v^2)\right)(u^2 + 2v^2)\, du\, dv$$

$$= \int_0^{\sqrt{5}} \int_0^{\sqrt{5}} (u^2 - 2v^2)(u^2 + 2v^2)\, du\, dv$$

$$= \int_0^{\sqrt{5}} \int_0^{\sqrt{5}} u^4 - 4v^4\, du\, dv$$

$$= \int_0^{\sqrt{5}} \int_0^{\sqrt{5}} u^4\, du\, dv \quad - 4 \int_0^{\sqrt{5}} \int_0^{\sqrt{5}} v^4\, du\, dv$$

$$= \sqrt{5} \int_0^{\sqrt{5}} u^4\, du \quad - 4\sqrt{5} \int_0^{\sqrt{5}} v^4\, dv$$

$$= \sqrt{5} \left(\frac{1}{5} u^5 \Big|_{u=0}^{u=\sqrt{5}} \right) - 4\sqrt{5} \left(\frac{1}{5} v^5 \Big|_{v=0}^{v=\sqrt{5}} \right)$$

$$= \sqrt{5} \left(\frac{1}{5} \cdot 25\sqrt{5} \right) - 4\sqrt{5} \left(\frac{1}{5} \cdot 25\sqrt{5} \right)$$

$$= 25 - 100$$

$$= -75$$

9. Compute the flux of $\mathbf{F} = (y, -x, 1)$ outward through the upper hemisphere of the sphere $x^2+y^2+z^2 = 4$.

Let $\Phi = (2\sin\varphi\cos\theta,\ 2\sin\varphi\sin\theta,\ 2\cos\varphi) \qquad 0 \le \theta \le 2\pi,\ 0 \le \varphi \le \frac{\pi}{2}$

$$\frac{\partial\Phi}{\partial\theta} = 2(-\sin\varphi\sin\theta,\ \sin\varphi\cos\theta,\ 0)$$

$$\frac{\partial\Phi}{\partial\varphi} = 2(\cos\varphi\cos\theta,\ \cos\varphi\sin\theta,\ -\sin\varphi)$$

$$\vec{n}_1 = \frac{\partial\Phi}{\partial\theta} \times \frac{\partial\Phi}{\partial\varphi} = 4\begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ -\sin\varphi\sin\theta & \sin\varphi\cos\theta & 0 \\ \cos\varphi\cos\theta & \cos\varphi\sin\theta & -\sin\varphi \end{vmatrix}$$

$$= 4\left(-\sin^2\varphi\cos\theta,\ -\sin^2\varphi\sin\theta,\ -\sin\varphi\cos\varphi\right)$$

$$= -4\sin\varphi\left(\sin\varphi\cos\theta,\ \sin\varphi\sin\theta,\ \cos\varphi\right)$$

This points inward, so take $\vec{n} = -\vec{n}_1 = 4\sin\varphi(\sin\varphi\cos\theta, \sin\varphi\sin\theta, \cos\varphi)$

$$\iint_S \vec{F}\cdot d\vec{S} = \iint_D (2\sin\varphi\sin\theta,\ -2\sin\varphi\cos\theta,\ 1)\cdot(\sin\varphi\cos\theta, \sin\varphi\sin\theta, \cos\varphi)\cdot 4\sin\varphi\, d\varphi\, d\theta$$

$$= \int_0^{2\pi}\int_0^{\frac{\pi}{2}} \left(2\sin^2\varphi\sin\theta\cos\theta - 2\sin^2\varphi\sin\theta\cos\theta + \cos\varphi\right)(4\sin\varphi)\, d\varphi\, d\theta$$

$$= \int_0^{2\pi}\int_0^{\frac{\pi}{2}} 4\sin\varphi\cos\varphi\, d\varphi\, d\theta$$

$$= 2\pi\int_0^{\frac{\pi}{2}} 4\sin\varphi\cos\varphi\, d\varphi$$

$$= 2\pi\left(2\sin^2\varphi\right)\Big|_{\varphi=0}^{\varphi=\frac{\pi}{2}}$$

$$= 2\pi(2-0)$$

$$= 4\pi$$

10. Let S be the torus parameterized by $\Phi = ((1+\cos\phi)\cos\theta, (1+\cos\phi)\sin\theta, \sin\phi)$, $0 \leq \phi, \theta \leq 2\pi$, oriented with an outward pointing normal. Let $\mathbf{F} = (x^3 \sin y, e^{x+z}\cos y, z^3)$. Compute $\iint_S curl(\mathbf{F}) \cdot d\mathbf{S}$.

S is closed. By Stokes Thm, $\iint_S \text{curl}\,\vec{F} \cdot d\vec{S} = 0$

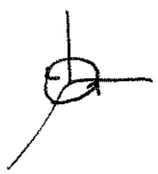
11. Let $\mathbf{G}$ be a vector field and $\mathbf{F} = curl(G)$. If $\iint_S \mathbf{F}\cdot d\mathbf{S} = 2\pi$ for S the unit disk in the xy-plane (oriented with upward pointing normal), compute the following, justifying your answers:

(a) The outward flux of $\mathbf{F}$ through the upper hemisphere of the unit sphere $x^2+y^2+z^2=1$

(b) The outward flux of $\mathbf{F}$ through the lower hemisphere of the unit sphere $x^2+y^2+z^2=1$

(c) The inward flux of $\mathbf{F}$ through the portion of the (half) cone $z = 1-\sqrt{x^2+y^2}$ lying above the xy-plane

(d) The inward flux of $\mathbf{F}$ through the portion of the (half) cone $z = \sqrt{x^2+y^2}-1$ lying below the xy-plane



Let C be unit circle in xy-plane oriented CCW.

$C = \partial S$

Stokes $\Rightarrow 2\pi = \iint_S \vec{F}\cdot d\vec{S} = \oint_C \vec{G}\cdot d\vec{s}$

(a) ∂(upper hemisphere, outward) $= C$

So $\iint \vec{F}\cdot d\vec{S} = \oint_C \vec{G}\cdot d\vec{s} = 2\pi$

(b) ∂(lower hemisphere, outward) $= -C$

$\iint \vec{F}\cdot d\vec{S} = \oint_{-C} \vec{G}\cdot d\vec{s} = -\oint_C \vec{G}\cdot d\vec{s} = -2\pi$

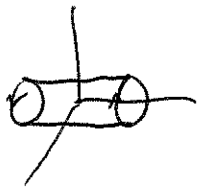
(c) ∂(up cone, inward) $= -C$

$\iint \vec{F}\cdot d\vec{S} = \oint_{-C} \vec{G}\cdot d\vec{s} = -\oint_C \vec{G}\cdot d\vec{s} = -2\pi$

(d) ∂(down cone, inward) $= C$

$\iint \vec{F}\cdot d\vec{S} = \oint_C \vec{G}\cdot d\vec{s} = 2\pi$

12. Let $\mathbf{G} = (x, xyz, yx + z)$ and $\mathbf{F} = curl(\mathbf{G})$. Compute the flux of $\mathbf{F}$ outward through the cylinder $x^2 + z^2 = 1$, $-2 \le y \le 2$.



$$\iint_S \vec{F} \cdot d\vec{S} = \iint_S \operatorname{curl} \vec{G} \cdot d\vec{S}$$

$$= \oint_{\partial S} \vec{G} \cdot d\vec{s} \qquad \text{(Stokes)}$$

$$= \oint_{r_1} \vec{G} \cdot d\vec{s} + \oint_{r_2} \vec{G} \cdot d\vec{s}$$

$r_1(t) = (\cos t, 2, \sin t)$

$r_2(t) = (\cos t, -2, -\sin t)$

$$= \int_0^{2\pi} (\cos t, 2\cos t \sin t, 2\cos t + \sin t) \cdot (-\sin t, 0, \cos t)\, dt$$
$$+ \int_0^{2\pi} (\cos t, 2\cos t \sin t, -2\cos t - \sin t)(-\sin t, 0, -\cos t)\, dt$$

$$= \int_0^{2\pi} -\cos t \sin t + 0 + 2\cos^2 t + \sin t \cos t\, dt$$
$$+ \int_0^{2\pi} -\cos t \sin t + 0 + 2\cos^2 t + \sin t \cos t\, dt$$

$$= \int_0^{2\pi} 2\cos^2 t\, dt + \int_0^{2\pi} 2\cos^2 t\, dt$$

$$= 2 \int_0^{2\pi} 2\cos^2 t\, dt$$

$$= 2 \cdot 2 \left(\tfrac{1}{2} t + \tfrac{1}{4} \sin 2t \right) \Big|_{t=0}^{t=2\pi}$$

$$= 4 (\pi + 0 - (0+0))$$

$$= 4\pi$$

$\sin^2(\theta) + \cos^2(\theta) = 1$
$\tan^2(\theta) + 1 = \sec^2(\theta)$
$\cot^2(\theta) + 1 = \csc^2(\theta)$

$\sin(2\theta) = 2\sin(\theta)\cos(\theta)$
$$\begin{aligned}\cos(2\theta) &= \cos^2(\theta) - \sin^2(\theta)\\ &= 2\cos^2(\theta) - 1\\ &= 1 - 2\sin^2(\theta)\end{aligned}$$
$\tan(2\theta) = \frac{2\tan(\theta)}{1-\tan^2(\theta)}$

$\sin(\frac{\theta}{2}) = \pm\sqrt{\frac{1-\cos\theta}{2}}$
$\cos(\frac{\theta}{2}) = \pm\sqrt{\frac{1+\cos\theta}{2}}$

$\sin(0) = 0$
$\cos(0) = 1$
$\sin(\frac{\pi}{6}) = \frac{1}{2}$
$\cos(\frac{\pi}{6}) = \frac{\sqrt{3}}{2}$
$\sin(\frac{\pi}{4}) = \frac{\sqrt{2}}{2}$
$\cos(\frac{\pi}{4}) = \frac{\sqrt{2}}{2}$
$\sin(\frac{\pi}{3}) = \frac{\sqrt{3}}{2}$
$\cos(\frac{\pi}{3}) = \frac{1}{2}$
$\sin(\frac{\pi}{2}) = 1$
$\cos(\frac{\pi}{2}) = 0$

$$\det\begin{pmatrix} a & b\\ c & d\end{pmatrix} = ad - bc$$
$$\det\begin{pmatrix} a & b & c\\ d & e & f\\ g & h & i\end{pmatrix} = a\det\begin{pmatrix} e & f\\ h & i\end{pmatrix} - b\begin{pmatrix} d & f\\ g & i\end{pmatrix} + c\det\begin{pmatrix} d & e\\ g & h\end{pmatrix}$$

$\lim_{x\to 0}\frac{\sin x}{x} = 1$
$\lim_{x\to 0}\frac{1-\cos x}{x} = 0$
$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e$

$\frac{d}{dx}(\sin x) = \cos x$
$\frac{d}{dx}(\cos x) = -\sin x$
$\frac{d}{dx}(\tan x) = \sec^2 x$
$\frac{d}{dx}(\sec x) = \sec x\tan x$
$\frac{d}{dx}(\csc x) = -\csc x\cos x$
$\frac{d}{dx}(\cot x) = -\csc^2 x$

$\int\sin x\ dx = -\cos x + C$
$\int\cos x\ dx = \sin x + C$
$\int\tan x\ dx = \ln|\sec x| + C$
$\int\cot x\ dx = \ln|\sin x| + C$
$\int\sec x\ dx = \ln|\sec x + \tan x| + C$
$\int\csc x\ dx = \ln|\csc x - \cot x| + C$

$\int\frac{1}{\sqrt{1-x^2}}\ dx = \sin^{-1}x + C$

$\int\frac{1}{1+x^2}\ dx = \tan^{-1}x + C$

$\int u\ dv = uv - \int v\ du$

$\int\sin^2 x\ dx = \frac{1}{2}x - \frac{1}{4}\sin 2x + C$
$\int\cos^2 x\ dx = \frac{1}{2}x + \frac{1}{4}\sin 2x + C$
$\int\sin^n x\ dx = -\frac{1}{n}\sin^{n-1}x\cos x + \frac{n-1}{n}\int\sin^{n-2}x\ dx$
$\int\cos^n x\ dx = \frac{1}{2}\cos^{n-1}x\sin x + \frac{n-1}{n}\int\cos^{n-2}x\ dx$